Red-Blue Pebbling number of some graphs

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Abstract: **Red-Blue pebble game was introduced by Hong and Kung [1]. Given any DAG, $G = (V, E)$, a configuration C on G is a function $C : V(G) \rightarrow \{R, B, O\}$ where $V(G)$ can be partitioned into $V_1(G), V_2(G)$ and $V_3(G)$ in such a way that $V_1(G)$ comprises of just the vertices having red pebbles (R), $V_2(G)$ is just those have blue pebbles (B), and $V_3(G)$ is the empty set that is vertices have no pebbles. Define the size $|C|$ of a configuration C to be the total number of pebbles, that is $|C| = \sum_{v \in V(G)} C(V)$. In this paper, we determine the $min|C|$ pebbles used in the completion of Red-Blue pebble game for different Directed Acyclic graphs (DAG) such as r-pyramid, and complete r-partite graphs.**

Keywords: **Red-Blue pebble game, different Directed Acyclic graphs.**

1. INTRODUCTION

Here, a graph G is a directed acyclic graph with vertices $V(G)$ and edge set $E(G)$ is the set of ordered pair of vertices (v_i, v_j) such that $i \neq j$ and $v_i, v_j \in V(G)$. We say that a vertex v_i is a direct predecessor of a vertex v_j if there is a directed edge from v_i to v_j and the edge $v_i v_j$ is called the incoming edge of v_j and outgoing edge of v_i. A vertex in a DAG, with no incoming edges is called a source vertex and a vertex with no outgoing edges is called a target vertex.

We assume that the set of source vertices are different from that of target vertices. Blue pebbles represent data that is stored in slow memory and red pebbles represent data that is stored in fast memory [2]. There are no restrictions on the number of blue pebbles that can reside on G at any given time, but we can never have more than S red pebbles on DAG $G = (V, E)$ where S represents the size of the fast memory. In the begining there is a blue pebble on each of the source vertices called as initial configuration and the game is completed when we have a blue pebble on each of the target vertices called as final configuration.

A pebbling move is an ordered pair of configurations, the second of which follows from the first according to certain rules.[3]

Pebbling move 1 that is $M1$ is defined as "A red pebble may be placed on any vertex that has a blue pebble" i.e.,) Replace a blue pebble by a red pebble.

Pebbling move 2 that is $M2$ is defined as "A blue pebble may be placed on any vertex that has a red pebble" that is replace a red pebble by a blue pebble.

Pebbling move 3 that is $M3$ is defined as "'Place a red pebble on a vertex for which all immediate predecessors are carrying a red pebble"' (when the vertex has already a blue pebble,

we replace it).

Pebbling move 4 that is $M4$ is 'A pebble red or blue may be removed from any vertex'. Since the size of our fast memory is limited, we will have to apply moves $M2$ and $M4$ sometimes.

A pebbling strategy is a sequence of the pebbling move $M1, M2, M3$ and $M4$ on the vertices of a DAG, which results in the completion of red blue pebble game.

Pebbling moves $M1|M2$ represent data movements that consume much time and energy, so we want to minimize their number. Each application of $M1$ or $M2$ counts as one $M1|M2$.

For any DAG G, we define the red-blue pebbling number $RB(G)$ is the smallest positive integer m such that m pebbles (blue, red) are involved in the completion of red-blue pebble game with minimal $M1|M2$ steps.

2. RED-BLUE PEBBLING NUMBER

Definition 2.1. A directed graph $G = (V, E)$ is called a layered graph with n levels if V can be written as a disjoint union of non-empty sets $V_1, V_2, \ldots, V_n$ such that for all $e = (u, v) \in E$, there exists i such that $u \in V_i$ and $v \in V_{i+1}$.

Definition 2.2. An r-pyramid of height n, $P_r(n)$ is a graph $(V_r(n), E_r(n))$ with the following

properties:

1. $P_r(n) = (V_r(n), E_r(n))$ is a layered graph with height n and $n + 1$ levels. Here

$V_r(n) = V_1 \cup V_2 \cup \ldots \cup V_{n+1}$, vi is the set of vertices on level i, $1 \le i \le n + 1$, and $Er(n)$

are the edges.

2. v_i has $n_r(i) = (r - 1) * (i - 1) + 1$ vertices labeled $v(i, 1), \cdots, v(i, n_r(i))$.

3. vertex $v(i, j)$ has r incoming edges from vertices $v(i + 1, j), v(i + 1, j + 1), \cdots, v(i + 1, j + r - 1)$.

4. There are no other edges in $P_r(n)$.

For any vertex v in the r-pyramid the subgraph rooted at v is a star $K_{1,r}$. Following is the list of $K_{1,r}$ in height n of r-pyramid and their corresponding root, vertices and in $(K_{1,r})_{(i,j)}$, (i, j) refers to $i = 1, 2, 3, \ldots$(position of $K_{1,r}$) and $j = 1, 2, 3, \ldots, n$ (of height).

Table 1

Height	Number of $K_{1,r}$	Name	Root vertex
1	1	$(K_{1,r})_{(1,1)}$	t
2	r	$(K_{1,r})_{(1,2)}$ $(K_{1,r})_{(2,2)}$ … $(K_{1,r})_{(r,2)}$	x_{12} x_{22} … x_{r2}
3	$2r - 1$	$(K_{1,r})_{(1,3)}$ $(K_{1,r})_{(2,3)}$ … $(K_{1,r})_{(2r-1,3)}$	x_{13} x_{23} … $x_{(2r-1)3}$
4	$3r - 2$	$(K_{1,r})_{(1,4)}$ $(K_{1,r})_{(2,4)}$ … $(K_{1,r})_{(3r-2,4)}$	x_{14} x_{24} … $x_{(3r-2)4}$
…	…	…	…
$n - 2$	$(n - 3)r - (n - 4)$	$(K_{1,r})_{(1,n-2)}$ $(K_{1,r})_{(2,n-2)}$ … $(K_{1,r})_{((n-3)r-(n-5),n-2)}$ $(K_{1,r})_{((n-3)r-(n-4),n-2)}$	$x_{1(n-2)}$ $x_{2(n-2)}$ … $x_{((n-3)r-(n-5))(n-2)}$ $x_{((n-3)r-(n-4))(n-2)}$
$n - 1$	$(n - 2)r - (n - 3)$	$(K_{1,r})_{(1,n-1)}$ $(K_{1,r})_{(2,n-1)}$ …	$x_{1(n-1)}$ $x_{2(n-1)}$ …

		$(K_{1,r})_{((n-2)r-(n-4),n-1)}$ $(K_{1,r})_{((n-2)r-(n-3),n-1)}$	$x_{((n-2)r-(n-4))(n-1)}$ $x_{((n-2)r-(n-3))(n-1)}$
n	$(n-1)r-(n-2)$	$(K_{1,r})_{(1,n)}$ $(K_{1,r})_{(2,n)}$ $\ldots$ $(K_{1,r})_{((n-1)r-(n-2),n)}$	x_{1n} x_{2n} $\ldots$ $x_{((n-1)r-(n-2))n}$

Any r-pyramid has $nr-(n-1)$ source vertices $s_1, s_2, \ldots, s_{nr-(n-1)}$ and let the target vertex be t. Also, level $(n+1)$ has $nr-(n-1)$ vertices. i.e., level $1,2,3,4,\ldots,n$ has $1, r, 2r-1, 3r-2, \ldots, (n-1)r-(n-2)$ vertices respectively.

Theorem 2.1. For any DAG, $G = P_r(n)$, a r-pyramid of height n, $r \geq 2, n \geq 2$, the red blue pebbling number $RB(G) = n(r-1)+1$.

Proof. By placing blue pebble on each of the source vertices $s_1, s_2, \ldots, s_{nr-(n-1)}$.

Consider$(K_{1,r})_{(1,n)}$.

By $M1$, replace the blue pebble at each of the source vertices $s_1, s_2, \ldots, s_r$ by red pebble. By $M3$, an additional red pebble be placed on x_{1n}. By $M4$, remove the red pebble from s_1 and place it on s_{r+1}.

Replacing red pebble by blue pebble on x_{1n} by $M2$.

Now consider $(K_{1,r})_{(2,n)}$.

By $M3$, place the freed red pebble from x_{1n} on x_{2n}. By $M4$, free the pebble at s_2 and placethe freed pebble on s_{r+2}.

Consider $(K_{1,r})_{(3,n)}$.

By $M3$, place an additional red pebble on x_{3n}. By $M4$, free the pebble at s_3 and place the freed pebble on s_{r+3}. By $M3$, place one more additional red pebble on x_{4n}. Continuing like this until $x_{((n-1)r-(n-2))n}$ is pebbled with red pebble.

Among $(n-1)r-(n-2)$ vertices of level n, x_{1n} is pebbled with blue pebble and each of $x_{2n}, x_{3n}, \ldots, x_{((n-1)r-(n-2))n}$ pebbled with red pebbles and source vertices $nr-(n-r), nr-(n-r-1), \cdots, nr-(n-1)$ each have a red pebble.

Consider $(K_{1,r})_{((n-2)r-(n-3),n-1)}$.

Let us free the red pebble at $s_{nr-(n-1)}$ and place this freed red pebble in $x_{((n-2)r-(n-3))(n-1)}$.

Free the red pebble at s_{nr-n} and place this freed red pebble in $x_{((n-2)r-(n-2))(n-1)}$. And proceeding like this, free the red pebble at $s_{nr-(n-r)}$and place this freed red pebble in x_{11} by replacing the blue pebble by $M1$.

In level $(n-1)$, there are $(n-2)r-(n-3)$ root vertices, hence there exist $(n-2)r-(n-3)$, $K_{1,r}$ exist at height $(n-1)$. Free the red pebbles from $(r-1)$ source vertices, and place this freed red pebble in $r-1$ vertices (from $(n-2)r-(n-3)$ vertices). From right to left, from level n vertices red pebbles are being moved to level $n-1$ vertices, and from level $(n-1)$ vertices red pebbles are being shifted to level $(n-2)$ vertices and so on, until target vertex t is reached.

By $M2$, replace red by blue in the target vertex t.So number of blue pebbles used $= nr-(n-1)$.

Number of red pebbles used $= r+(n-1)r-(n-2)-1 = n(r-1)+1$.

Number of $M1|M2$ moves $= nr-(n-1)+1+1 = n(r-1)+3$.

Red blue pebble game strategies are as follows:

- Place a blue pebble on each of the source vertices $s_1, s_2, \ldots, s_r, s_{r+1}, s_{r+2}, \ldots, s_{nr-(n-1)}$.
- $M1(s_1), M1(s_2), \ldots, M1(s_r)$
- $M3(x_{1n})$
- $M4(s_1)$

- $M1(s_{r+1})$
- $M2(x_{1n})$
- $M3(x_{2n})$
- $M4(s_2)$
- $M1(s_{r+2})$
- $M3(x_{2n})$

….

- $M1(s_{nr-(n-1)})$
- $M3(x_{((n-1)r-(n-2))n})$
- $M4(s_{nr-(n-1)})$
- $M1(x_{1n})$
- $M4(s_{nr-(n-2)})$
- $M3(x_{((n-2)r-(n-3))(n-1)})$
- $M4(s_{nr-(n-3)})$
- $M3\left(x_{((n-2)r-(n-2))(n-1)}\right)$

….

- $M4(s_{nr-(n-r)})$
- $M3(x_{((n-2)r-(n-1))(n-1)})$

….

- $M3(x_{((n-2)r-(n-r+1))(n-1)})$
- $M4(x_{((n-1)r-(n-2))n})$
- $M3(x_{((n-2)r-(n-r+2))(n-1)})$
- $M4(x_{((n-1)r-(n-1))n})$
- $M3(x_{((n-2)r-(n-r+3))(n-1)})$

and so on until target vertex t is pebbled.

Definition 2.3. A complete r-partite graph is a r-partite graph (i.e.,) a set of graph vertices decomposed into r disjoint sets such that no two graph vertices within the same set are adjacent) such that every pair of graph vertices in the r sets are adjacent. If there are $p, q, \dots, t$ graph vertices in the r sets, the complete r-partite graph is denoted by $K_{p,q,\dots,t}$.

Theorem 2.2. For any DAG, $G = K_{2,2,1}$, red-blue pebbling number $RB(G) = 5$.

Proof. Let $V(K_{2,2,1}) = \{s_1, s_2, y_1, y_2, t\}$ *and* $E(K_{2,2,1}) = \{s_1y_1, s_1y_2, s_2y_1, s_2y_2, y_1t, y_2t\}$.

Source vertices s_1, s_2 and Target vertex t.

Pebbling strategies are as follows:

- By placing blue pebble on each of the source vertices s_1 and s_2.
- $M1(s_1) and M1(s_2)$
- $M3(y_1) and M3(y_2)$

- $M4(s_1)$
- $M3(t)$
- $M2(t)$

Number of red pebbles involved = 3.

Number of blue pebbles = 2.

So red-blue pebbling number $RB(G) = 5$. Number of $M1|M2$ moves =3.

Theorem 2.3. For any DAG, $G = K_{3,4,4,2}$, red blue pebbling number $RB(G) = 11$.

Proof. Let $V(K_{3,4,4,2}) = \{s_1, s_2, s_3, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, t_1, t_2\}$ *and*

$$E(K_{3,4,4,2}) = \{s_1x_1, s_1x_2, s_1x_3, s_1x_4, s_2x_1, s_2x_2, s_2x_3, s_2x_4, s_3x_1, s_3x_2, s_3x_3, s_3x_4, x_1y_1, x_1y_2, x_1y_3, x_1y_4, x_2y_1, x_2y_2, x_2y_3, x_2y_4, x_3y_1, x_3y_2, x_3y_3, x_3y_4, x_4y_1, x_4y_2, x_4y_3, x_4y_4, y_1t_1, y_2t_1, y_3t_1, y_4t_1, y_1t_2, y_2t_2, y_3t_2, y_4t_2\}.$$

Pebbling strategies are given as follows:

- Blue pebbles are placed on each of the source vertices s_1, s_2 and s_3.
- $M1(s_1), M1(s_2), M1(s_3)$
- $M3(x_1), M3(x_2), M3(x_3), M3(x_4)$
- $M4(s_1), M3(y_1)$
- $M4(s_2), M3(y_2)$
- $M4(s_3), M3(y_3)$
- $M3(y_4)$
- $M4(x_1), M3(t_1)$
- $M4(x_2), M3(t_2)$
- $M2(t_1), M2(t_2)$

Number of red pebbles = 11.

Number of blue pebbles = 3.

Number of $M1|M2$ steps =5.

Red Blue pebbling number $RB(G) = 14$.

Theorem 2.4. For any DAG, $G = K_{3,3,3,3}$, red blue pebbling number $RB(G) = 9$.

Proof. Let $V(K_{3,3,3,3}) = \{s_1, s_2, s_3, x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2, t_3\}$ *and*

$$E(K_{3,3,3,3}) = \{s_1x_1, s_1x_2, s_1x_3, s_2x_1, s_2x_2, s_2x_3, s_3x_1, s_3x_2, s_3x_3, x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3, y_1t_1, y_1t_2, y_1t_3, y_2t_1, y_2t_2, y_2t_3, y_3t_1, y_3t_2, y_3t_3\}.$$

Pebbling strategies are as follows:

- By placing blue pebbles on each of the source vertices s_1, s_2 and s_3.
- $M1(s_1), M1(s_2), M1(s_3)$
- $M3(x_1), M3(x_2), M3(x_3)$
- $M4(s_1), M3(y_1)$

- $M4(s_2), M3(y_2)$
- $M4(s_3), M3(y_3)$
- $M4(x_1), M3(t_1)$
- $M4(x_2), M3(t_2)$
- $M4(x_3), M3(t_3)$
- $M2(t_1), M2(t_2), M2(t_3)$

Number of red pebbles = 6.

Number of blue pebbles = 3.

Number of $M1|M2$ steps =6.

Red Blue pebbling number RB(G) = 9.

Theorem 2.5. For any DAG, $G = K_{4,2,4}$, red blue pebbling number $RB(G) = 10$.

Proof. Let $V(K_{4,2,4}) = \{s_1, s_2, s_3, s_4, x_1, x_2, t_1, t_2, t_3, t_4\}$ *and*

$E(K_{4,2,4}) = \{s_4 x_1, s_4 x_2, s_3 x_1, s_3 x_2, s_2 x_1, s_2 x_2, s_1 x_1, s_1 x_2, x_1 t_1, x_1 t_2, x_1 t_3, x_1 t_4, x_2 t_1, x_2 t_2, x_2 t_3, x_2 t_4\}$.

Pebbling strategies are as follows:

- By placing blue pebble on each of the source vertices s_1, s_2, $s_3$and s_4.
- $M1(s_1), M1(s_2), M1(s_3), M1(s_4)$
- $M3(x_1), M3(x_2)$
- $M4(s_1), M3(t_1)$
- $M4(s_2), M3(t_2)$
- $M4(s_3), M3(t_3)$
- $M4(s_4), M3(t_4)$
- $M2(t_1), M2(t_2), M2(t_3), M2(t_4)$

Number of red pebbles = 6.

Number of blue pebbles = 4.

Minimum number of $M1|M2$ steps =8.

Red Blue pebbling number $RB(G) = 10$.

Theorem 2.6. For any DAG, $G = K_{3,3,3,1}$, red blue pebbling number $RB(G) = 9$.

Proof. Let $V(K_{3,3,3,1}) = \{s_1, s_2, s_3, x_1, x_2, x_3, y_1, y_2, y_3, t\}$ *and*

$E(K_{3,3,3,1}) = \{s_1 x_1, s_2 x_2, s_3 x_3, x_1 y_1, x_1 y_2, x_1 y_3, x_2 y_1, x_2 y_2, x_2 y_3, x_3 y_1, x_3 y_2, x_3 y_3, y_1 t, y_2 t, y_3 t\}$.

Pebbling strategies are as follows:

- By placing a blue pebble on each of the source vertices s_1, s_2 and s_3.
- $M1(s_1), M1(s_2), M1(s_3)$
- $M3(x_1), M3(x_2), M3(x_3)$
- $M4(s_1), M3(y_1)$
- $M4(s_2), M3(y_2)$
- $M4(s_3), M3(y_3)$

- $M4(x_1), M3(t)$
- $M2(t)$

Number of red pebbles = 6.

Number of blue pebbles = 3.

Minimum number of $M1|M2$ steps =4.

Red Blue pebbling number $RB(G) = 9$.

Theorem 2.7. For any DAG, $G = K_{n,m,s,1}$, red-blue pebbling number $RB(G) = 2n + xm$.

Proof. Let $V(K_{n,m,s,1}) = \{s_1, s_2, \dots, s_n, b_{11}, b_{12}, \cdots, b_{1m}, b_{21}, b_{22}, \cdots, b_{2m},$

$$b_{x1}, b_{x2}, \cdots, b_{xm}, z_1, z_2, \dots, z_x, t\} \text{ and}$$

$$E(K_{n,m,s,1}) = \{s_1 b_{11}, s_1 b_{12}, \cdots, s_1 b_{xm}, s_2 b_{11}, \cdots, s_2 b_{xm}, \cdots,$$

$$s_n b_{11}, s_n b_{12}, \cdots, s_n b_{xm}, b_{11} z_1, b_{12} z_1, \cdots, b_{1m} z_1,$$

$$b_{21} z_2, b_{22} z_2, \cdots, b_{2m} z_2, \cdots, b_{x1} z_x, b_{x2} z_x, \cdots, b_{xm} z_x, z_1 t, z_2 t, \cdots, z_x t\}.$$

Number of Red pebbles required = $n + \underbrace{m + m + \ldots + m}_{x\ times} = n + xm.$

Number of blue pebbles required = n.

Red blue pebbling number $RB(G) = 2n + xm$.

Minimum number of $M1|M2$ steps $= n + 1$

3. CONCLUSION

In this paper, we find the Red-Blue pebbling number of r-pyramid, and complete r-partite graphs of $K_{2,2,1}$,$K_{3,4,4,2}$,$K_{3,3,3,3}$,$K_{4,2,4}$,$K_{3,3,3,1}$,$K_{n,m,s,1}$. To find the Red-Blue pebbling number of $K_{s_1, s_2, \dots, s_r}$ where $s_1 \geq s_2 \geq \cdots \geq s_r$ is an open problem.

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